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ON THE THEORY OF SHOCK WAVES IN THE DYNAMICS OF A REAL GAS

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Аннотация—В статье на основе уравнения состояния для реального газа рассмотрены вопросы скачков уплотнений и получены важнейшие соотношения для высоких и сверхвысоких областей давлений.

NOMENCLATURE

- w, gas or vapour velocity, m/s;
- v, specific volume, m³/kg;
- p, pressure, kg/m²;
- T, absolute temperature, °K;
- z, compressibility factor;
- $\mu_T = -\dfrac{p^2}{RT}\left(\dfrac{\partial v}{\partial p}\right)_T$, $\mu_p = \dfrac{p}{R}\left(\dfrac{\partial v}{\partial T}\right)_p$, deviation coefficients;
- R, gas constant, kg m/kg g;
- x, index of real-gas adiabatic curve ("temperature");
- A, heat equivalent of work (= 1/427 kcal/kg m);
- c_p, heat capacity at constant pressure, kcal/kg g;
- s, entropy, kcal/kg g;
- g, gravity acceleration, m/s²;
- ρ, density, kg s²/m⁴;
- k, index of real-gas adiabatic curve;
- a, sonic velocity in real gas, m/s;
- y, ratio of sonic velocity in real gas to that in ideal gas;
- a_{cr}, critical velocity of real gas, m/s;
- λ, velocity coefficient;
- n, exponent of real-gas adiabatic curve ("volume");
- i, enthalpy, kcal/kg;
- a_0, sonic velocity in stationary gas.

IN GAS dynamics much attention is paid to the problem of shock waves or attached shock. All the difficulties, caused by the impossibility of integrating the equations without introducing discontinuities, are avoided by the theory of shock waves [1]. The relationship between flow parameters in front of and behind a shock wave was expressed by Hugonio in terms of the relation referred to as the equation of the shock-wave adiabatic-curve. This relation is derived on the basis of an energy equation, the law of momentum change and an equation of state for an ideal gas. Moreover, because of the sharp increase in pressure during the shock wave there naturally takes place some essential deviation from the ideal-gas laws used to obtain the equation of the shock-wave adiabatic-curve. As an example, reference can be made to shock waves in expanding nozzles of steam turbines under variable operating conditions. The Hugonio equation of a shock-wave adiabatic-curve, applied to nozzles of modern high-pressure steam turbines, cannot therefore ensure reliable calculation results. In future, errors appearing from the usage of the Hugonio equation will be very considerable for turbines using working agents considerably deviating from ideal gases, for example, carbonic acid.

The above considerations make it necessary to derive a new shock-wave adiabatic-curve equation applied to real gases with arbitrary parameters.

First of all, start with the energy equation for an adiabatic process in differential form:

$$\mathrm{d}\frac{w^2}{2g} + v\,\mathrm{d}p = 0. \quad (1)$$

Making use of the equation of state for a real gas in the form

$$pv = zRT \tag{2}$$

and taking into account that the compressibility factor is a function of p and T, it may be written that $pv = f(p,T)$, then

$$\left(\frac{\partial f}{\partial p}\right)_T \mathrm{d}p = \left[\frac{\partial(pv)}{\partial p}\right]_T \mathrm{d}p = p\left(\frac{\partial v}{\partial p}\right)_T \mathrm{d}p + v\,\mathrm{d}p. \tag{3}$$

If we account for equation (3) equation (1) assumes the form:

$$\mathrm{d}\frac{w^2}{2g} + \left(\frac{\partial f}{\partial p}\right)_T \mathrm{d}p - p\left(\frac{\partial v}{\partial p}\right)_T \mathrm{d}p = 0. \tag{4}$$

For convenience of calculations for real gases the deviation coefficients [2] are introduced

$$\mu_T = -\frac{p^2}{RT}\left(\frac{\partial v}{\partial p}\right)_T \tag{5}$$

and

$$\mu_p = \frac{p}{R}\left(\frac{\partial v}{\partial T}\right)_p \tag{6}$$

μ_p being used to write the adiabatic-curve equation for a real gas in the following form:

$$\frac{T_1}{T_2} = \left(\frac{p_1}{p_2}\right)^{(x-1)/x} \tag{7}$$

when the form of the relation, conforming to an ideal gas, is preserved.

In equation (7) x, related to μ_p and heat capacity c_p, is determined by the expression

$$x = \frac{c_p}{c_p - AR\,.\,\mu_p}. \tag{8}$$

On the basis of equation (5) equation (4) takes the form

$$\mathrm{d}\frac{w^2}{2g} + \left(\frac{\partial f}{\partial p}\right)_T \mathrm{d}p + \frac{RT}{p}\,.\,\mu_T\,\mathrm{d}p = 0. \tag{9}$$

For an entropy differential it may be written that

$$\mathrm{d}S = \left(\frac{\partial S}{\partial T}\right)_p \mathrm{d}T + \left(\frac{\partial S}{\partial p}\right)_T \mathrm{d}p$$

or, bearing in mind equation (6)

$$c_p\frac{\mathrm{d}T}{T} = AR\mu_p\frac{\mathrm{d}p}{p}. \tag{10}$$

Solving simultaneously equations (10) and (9) we have

$$w_1^2 - w_2^2 = 2g\int_1^2\left[\left(\frac{\partial f}{\partial p}\right)_T \mathrm{d}p + \frac{\mu_T}{\mu_p}\cdot\frac{c_p}{A}\mathrm{d}T\right]. \tag{11}$$

Values in front of and behind a shock wave will be denoted by suffixes 1 and 2.

Now transform the left-hand side of equation (11) proceeding from the momentum law, according to which, for an arbitrary state equation, it is obtainable:

$$p_2 - p_1 = \rho w(w_1 - w_2). \tag{12}$$

Multiplying both parts of equation (12) by $(w_1 + w_2)$, we have

$$(w_1 - w_2)(w_1 + w_2) = (p_2 - p_1)\frac{w_1 + w_2}{\rho w}$$

or taking into account the continuity equation

$$\rho_1 w_2 = \rho_2 w_2 \tag{13}$$

we obtain

$$w_1^2 - w_2^2 = (p_2 - p_1)\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right). \tag{14}$$

From equations (11) and (14), having regard to (8) it follows

$$2g\int_1^2\left[\left(\frac{\partial f}{\partial p}\right)_T \mathrm{d}p + \mu_T\,.\,\frac{x}{x-1}R\,\mathrm{d}T\right] = (p_2 - p_1)\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right). \tag{15}$$

In [2] it is shown that during integration quantities μ_T and $x/(x-1)$ may be considered constant and in this case the integral mean values may be replaced by the arithmetic mean values.

Henceforward, the mean value of the coefficient μ_T will be designated $\bar{\mu}_T$, and for convenience, the multiplier $x/(x-1)$ is preserved without an upper bar but it is however considered as an average quantity within integration limits.

Thus, integrating equation (15) with regard for (2), we have

$$2g\bar{\mu}_T \cdot \frac{x}{x-1} R(T_2 - T_1) - 2gRT_1(z_1 - z_2)_T = (p_2 - p_1)\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right). \quad (16)$$

The compressibility factors z_1 and z_2 being at different pressures but at one and the same temperature (T_1).

Furthermore, since according to equation (2)

$$T_1 = \frac{p_1}{g\rho_1 z_1 R} \quad \text{and} \quad T_2 = \frac{p_2}{g\rho_2 z_2 R}$$

equation (18) may be presented as

$$\frac{2x}{x-1}\left\{\frac{\bar{\mu}_T}{z_2} \cdot \frac{p_2}{\rho_2} - \left[\frac{\bar{\mu}_T}{z_1} - \frac{1}{z_1}(z_1 - z_2)\frac{x-1}{x}\right] \cdot \frac{p_1}{\rho_1}\right\} = (p_2 - p_1)\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right). \quad (17)$$

After transformation, we multiply both parts of the latter equation by ρ_2/p_1 and obtain

$$\frac{p_2}{p_1} = \left(\left\{2\frac{x}{z_1}\left[\bar{\mu}_T - (z_1 - z_2)_T \cdot \frac{x-1}{x}\right] - x + 1\right\} \cdot (x-1)^{-1} \cdot \frac{\rho_2}{\rho_1} - 1\right) \times \left\{\left[x\left(2 \cdot \frac{\bar{\mu}_T}{z_2} - 1\right) + 1\right] \cdot (x-1)^{-1} - \frac{\rho_2}{\rho_1}\right\}^{-1}. \quad (18)$$

Thus, the equation of the shock-wave adiabatic-curve applied to a real gas is obtained. The expression for the ratio of densities is presented as follows:

$$\frac{\rho_2}{\rho_1} = \left\{\left[x\left(2\frac{\bar{\mu}_T}{z_2} - 1\right) + 1\right] \cdot (x-1)^{-1} + \frac{p_1}{p_2}\right\} \times \left(\left\{2\frac{x}{z_1}\left[\bar{\mu}_T - (z_1 - z_2)_T \frac{x-1}{x}\right] - x + 1\right\}(x-1)^{-1}\frac{p_1}{p_2} + 1\right). \quad (19)$$

From the state equation for a real gas we obtain the following for the temperature ratio

$$\frac{T_2}{T_1} = \frac{p_2}{p_1} \cdot \frac{\rho_1}{\rho_2} \cdot \frac{z_1}{z_2}.$$

The relations of temperatures and densities in a shock wave are connected by

$$\frac{T_2}{T_1} + \frac{z_1}{z_2} \cdot \left(\left\{2\frac{x}{z_1}\left[\bar{\mu}_T - (z_1 - z_2)_T \frac{x-1}{x}\right] - x + 1\right\} \cdot (x-1)^{-1} - \frac{\rho_1}{\rho_2}\right) \times \left\{\left[x\left(2\frac{\bar{\mu}_T}{z_2} - 1\right) + 1\right] \cdot (x-1)^{-1} - \frac{\rho_2}{\rho_1}\right\}^{-1}. \quad (20)$$

Equations (18), (19) and (20) obtained for a real gas easily assume the form, usual in the shock-wave theory, in application to an ideal gas when passing to the condition $\bar{\mu}_T = z_1 = z_2 = 1$ and $x = k$. In this case the following well-known relations are obtained

$$\left(\frac{p_2}{p_1}\right)_{id} = \left[\frac{k+1}{k-1}\left(\frac{\rho_2}{\rho_1}\right)_{id} - 1\right] \cdot \left[\frac{k+1}{k-1} - \left(\frac{\rho_2}{\rho_1}\right)_{id}\right]^{-1}, \quad (21)$$

$$\left(\frac{\rho_2}{\rho_1}\right)_{id} = \left[\frac{k+1}{k-1} + \left(\frac{p_1}{p_2}\right)_{id}\right] \cdot \left[\frac{k+1}{k-1}\left(\frac{p_1}{p_2}\right)_{id} + 1\right]^{-1} \quad (22)$$

$$\left(\frac{T_2}{T_1}\right)_{id} = \left[\frac{k+1}{k-1} - \left(\frac{\rho_1}{\rho_2}\right)_{id}\right]\left[\frac{k+1}{k-1} - \left(\frac{\rho_2}{\rho_1}\right)_{id}\right]^{-1}. \quad (23)$$

With regard to the employment of equations (18), (19) and (20) it is necessary to determine first of all the numerical values of $\bar{\mu}_T, z_1, z_2$ and x. It has already been mentioned that the integral mean of $\bar{\mu}_T$ may be replaced by the arithmetic mean, i.e.

$$\bar{\mu}_T = \tfrac{1}{2} \times [(\mu_T)_1 + (\mu_T)_2].$$

The formula for μ_T is obtained from expression (5) in which the partial derivative is determined by differentiating equation (2). As a result, the value of μ_T assumes the form [2]

$$\mu_T = z - p\left(\frac{\partial z}{\partial p}\right)_T.$$

In a similar fashion from equation (7) by differentiating equation (2) we obtain

$$\mu_p = z + T\left(\frac{\partial z}{\partial T}\right)_p \text{ [1].}$$

Partial derivatives

$$\left(\frac{\partial z}{\partial p}\right)_T \text{ and } \left(\frac{\partial z}{\partial T}\right)_p$$

are determined by graphical differentiation.

To determine both $\bar{\mu}_T$, z_1, z_2 and x according to formula (8) where μ_p enters, it is necessary to possess initial and final data on pressures and temperatures, essential for using a compressibility diagram. Therefore, to the first approximation these values are defined from equations (21), (22) and (23) applied to an ideal gas. Then, upon calculation of the above coefficients we turn to equations (18), (19) and (20), using thus the method of successive approximations. To find the heat capacity c_p, entering into equation (8), it is possible to make use of the diagram given in [3], in which the relation between the correction $c_p - c_{p_0}$ and the above parameters is presented.

When using the shock-wave adiabatic-curve equation applied to water vapor, the coefficients μ_T, μ_p, z_1 and z_2 are found from the tables for superheated steam [4]. The compressibility factors are determined as

$$z_1 = \frac{p_1 v_1}{RT_1} \text{ and } z_2 = \frac{p_2 v_2}{RT_2}$$

the parameters p, v and T being tabular. The same tabular data may be used when determining the partial derivatives $(\partial z/\partial p)_T$ and $(\partial z/\partial T)_p$ and, consequently, μ_T and μ_p. The heat capacity c_p up to 300 atm may be taken from Vukalovich's tables [5] and above 300 atm, from the experimental data [6] over a wide range of pressures and temperatures.

The relations obtained may be also expressed in terms of the velocity coefficient in front of a shock wave, i.e. λ_1. Between the velocity coefficients in front of and behind a shock wave in a real gas there exists the same ratio as in the case of an ideal gas, viz.:

$$\lambda_1 \cdot \lambda_2 = 1. \tag{24}$$

We will derive relation (24) for a real gas. When a medium passes through a shock-wave front [7], the isentropic equation of the medium, considered as a real gas, is presented in just the same form as for the ideal gas, namely:

$$pv^{\bar{n}} = \text{const.} \tag{25}$$

It is necessary to mention that the integral mean of the isentropic index of a real gas $\bar{n}$ in equation (25) and that of the exponent x in equation (7) possess different significance. As was shown in [2], the quantity "$\bar{n}$" in equation (25) represents a "volumetric" exponent of an adiabatic curve. The writing of equation (7) has an advantage over equation (25), since the quantity $x/(x-1)$ is subjected to changes considerably less than "$\bar{n}$", and this is of great importance if integrating the corresponding expressions, for example, equation (15).

In writing down equation (25) we employ as qualitative a derivative as possible in principle for a real gas, to illustrate condition (24), the analytical value of the exponent "n" not being used in calculation equation.

According to equation (25) we can write

$$\frac{dp}{p} = -n\frac{dv}{v} \tag{26}$$

and taking into account that

$$a^2 = -gv^2\left(\frac{\partial p}{\partial v}\right)_s \tag{27}$$

where a is the sonic velocity, we have

$$a^2 = gnpv. \tag{28}$$

In [8] it was shown that

$$a^2 = y^2 kgRT \tag{29}$$

where according to the values of μ_T and μ_p

$$y^2 = z^2\left[k\left(\mu_T - \frac{x-1}{x}\mu_p\right)\right]^{-1}. \tag{30}$$

From equations (28) and (29) taking equation (2) into account the following is obtainable:

$$n = z\left(\mu_T - \frac{x-1}{x}\mu_p\right)^{-1}. \tag{31}$$

Thus, from the equation $di = Av\,dp$ and equation (26) it may be written that $di = -Anp\,dv$. Hence, taking into account equation

(27) upon integration and transformation we obtain

$$i_1 - i_2 = A \frac{\bar{n}}{\bar{n}-1} (p_1 v_1 - p_2 v_2). \quad (32)$$

On the basis of equation (1) we have

$$\frac{w_2^2 - w_1^2}{2g} = \frac{\bar{n}}{\bar{n}-1} (p_1 v_1 - p_2 v_2). \quad (33)$$

Hence, according to equation (28) the energy equation may be written as follows

$$\frac{a_1^2}{\bar{n}-1} + \frac{w_1^2}{2} = \frac{a_2^2}{\bar{n}-1} + \frac{w_2^2}{2} = \frac{a^2}{\bar{n}-1} + \frac{w^2}{2} = \frac{a_0^2}{\bar{n}-1} \quad (34)$$

where a_0 is the sonic velocity in stationary gas.

When $w = a_{cr}$ we have $a = a_{cr}$, then

$$a_0^2 = a_{cr}^2 \cdot \frac{\bar{n}+1}{2} \quad (35)$$

where a_{cr} is the critical velocity.

From equation (35) taking into account (28) we have

$$p_1 = \frac{\bar{n}+1}{2\bar{n}} \cdot \rho_1 \cdot a_{cr}^2 - \frac{\bar{n}-1}{2\bar{n}} \rho_1 w_1^2$$

$$p_2 = \frac{\bar{n}+1}{2\bar{n}} \cdot \rho_2 \cdot a_{cr}^2 - \frac{\bar{n}-1}{2\bar{n}} \rho_2 w_2^2 \quad (36)$$

or

$$p_2 - p_1 = a_{cr}^2 \frac{\bar{n}+1}{2\bar{n}} (\rho_2 - \rho_1) + \frac{\bar{n}-1}{2\bar{n}} (\rho_1 w_1^2 - \rho_2 w_2^2).$$

From equations (12) and (13) we have

$$p_2 - p_1 = \rho_1 w_1^2 - \rho_2 w_2^2. \quad (37)$$

Consequently, in this case equation (36) assumes the form

$$\frac{p_2 - p_1}{\rho_2 - \rho_1} = a_{cr}^2. \quad (38)$$

On the other hand, on the basis of equation (13) equation (37) may be also presented as in [9]

$$p_2 - p_1 = \rho_2 w_2 w_1 - \rho_1 w_1 w_2 = w_1 w_2 (\rho_2 - \rho_1). \quad (39)$$

From equations (38) and (39) we have

$$w_1 \cdot w_2 = a_{cr}^2 \quad (40)$$

or

$$\lambda_1 \cdot \lambda_2 = 1, \text{ i.e. condition (24) is proved.}$$

The relationship between λ_1 and λ_2 in front of and behind a shock wave so obtained makes it possible to present equations (18), (19) and (20) in terms of λ_1. Taking into account that

$$\frac{\rho_2}{\rho_1} = \frac{w_1}{w_2} = \frac{\lambda_1}{\lambda_2},$$

we obtain

$$\frac{p_2}{p_1} = \left(\left\{ 2 \frac{x}{z_1} \left[\bar{\mu}_T - (z_1 - z_2)_T \cdot \frac{x-1}{x} \right] - x + 1 \right\} (x-1)^{-1} \cdot \lambda_1^2 - 1 \right) \times \left\{ \left[x \left(2 \frac{\bar{\mu}_T}{z_2} - 1 \right) + 1 \right] \cdot (x-1)^{-1} - \lambda_1^2 \right\}^{-1} \quad (41)$$

$$\frac{\rho_2}{\rho_1} = \lambda_1^2 \quad (42)$$

$$\frac{T_2}{T_1} = \frac{z_1}{z_2} \left(\left\{ 2 \frac{x}{z_1} \left[\bar{\mu}_T - (z_1 - z_2)_T \cdot \frac{x-1}{x} \right] - x + 1 \right\} \times (x-1)^{-1} - \frac{1}{\lambda_1^2} \right) \left\{ \left[x \left(\frac{\bar{\mu}_T}{z_2} - 1 \right) + 1 \right] \cdot (x-1)^{-1} - \lambda_1^2 \right\}^{-1}. \quad (43)$$

It has been already mentioned that the expression $x/(x-1)$ is understood as the arithmetic mean, i.e.

$$\frac{x}{x-1} = \frac{1}{2} \left(\frac{x_1}{x_1 - 1} + \frac{x_2}{x_2 - 1} \right)$$

consequently,

$$x = \frac{x_1(2x_2 - 1) - x_2}{x_1 + x_2 - 2}. \quad (44)$$

In all our equations the quantity x is defined by relation (44) and to calculate both x_1 and x_2 it is necessary only to use equation (8).

When passing from a real to an ideal gas, for which $\bar{\mu}_T = z_1 = z_2 = 1$ and $x = k$, equations

(41) and (43) assume the form well-known in gas dynamics [10]

$$\left(\frac{p_2}{p_1}\right)_{id} = \left(\lambda_{id}^2 - \frac{k-1}{k+1}\right) \cdot \left(1 - \frac{k-1}{k+1} \cdot \lambda_{id}^2\right)^{-1} \quad (45)$$

$$\left(\frac{T_2}{T_1}\right)_{id} = \left(1 - \frac{k-1}{k+1} \cdot \frac{1}{\lambda_{id}^2}\right) \cdot \left(1 - \frac{k-1}{k+1} \cdot \lambda_{id}^2\right)^{-1} \quad (46)$$

In conclusion it should be noted that the ratios obtained for a direct shock-wave make it possible to study easily shock waves with the front shock-wave inclined to the direction of flow, i.e. to investigate oblique shock-waves.

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Abstract—In the present paper problems on shock waves are considered on the basis of the equation of state, and important relations for high and superhigh pressure regions are obtained.

Résumé—Cet article considère les ondes de choc à partir de l'équation d'état, et établit des relations importantes pour des régions de pressions élevées et très élevées.

Zusammenfassung—In der Arbeit werden Stosswellenprobleme auf der Basis der Zustandsgleichung betrachtet und wichtige Beziehungen für hohe und superhohe Druckbereiche erhalten.